

On Partial Smoothness, Tilt Stability and the \mathcal{VU} -Decomposition

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Abstract

Under the assumption of prox-regularity and the presence of a tilt stable local minimum we are able to show that a \mathcal{VU} like decomposition gives rise to the existence of a smooth manifold on which the function in question coincides locally with a smooth function.

1 Introduction

The study of substructure of nonsmooth functions has led to a enrichment of fundamental theory of nonsmooth functions [13, 11, 12, 15, 16, 17, 18]. Fundamental to this substructure is the presence of manifolds along which the restriction of the nonsmooth function exhibits some kind of smoothness. In the case of partially smooth function an axiomatic approach is used to describe the local structure that is observed in a number of important examples [16], [17]. In the theory of the \mathcal{U} -Lagrangian and the associated \mathcal{VU} decomposition the existence of a smooth manifold substructure is proven for some special classes of functions. In the extended theory the presence of so called “fast tracks” is assumed and these also give rise to similar manifold substructures. The \mathcal{U} -Lagrangian is reminiscent of a partial form of tilt minimisation and this observation has motivated this study. As fast tracks are designed to aid the design of efficient methods for the solution of nonsmooth minimization problems it seems appropriate to ask what additional structure does the existence of a tilt stable local minimum give to the study of the \mathcal{VU} decomposition [15]? This is the subject of the paper.

Denote the extended real by $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. If not otherwise stated we will consider a lower semi-continuous, extended real valued, prox-regular [21] function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$. Denote by $\partial_p f(\bar{x})$ the proximal subdifferential, which consists of all vectors z satisfying $f(x) \geq f(\bar{x}) + \langle z, x - \bar{x} \rangle - \frac{r}{2} \|x - \bar{x}\|^2$ in some neighbourhood of \bar{x} , for some $r \geq 0$, where $\|\cdot\|$ denotes the Euclidean norm. The limiting subdifferential [19, 23] at x is given by

$$\partial f(x) = \limsup_{x' \xrightarrow{f} x} \partial_p f(x') := \{z \mid \exists z_v \in \partial_p f(x_v), x_v \xrightarrow{f} x, \text{ with } z_v \rightarrow z\},$$

where $x' \xrightarrow{f} x$ means that $x' \rightarrow x$ and $f(x') \rightarrow f(x)$. The singular limiting subdifferential is given by

$$\begin{aligned} \partial^\infty f(x) &= \limsup_{x' \xrightarrow{f} x}^\infty \partial_p f(x') \\ &:= \{z \mid \exists z_v \in \partial_p f(x_v), x_v \xrightarrow{f} x, \text{ with } \lambda_v \downarrow 0 \text{ and } \lambda_v z_v \rightarrow z\}. \end{aligned}$$

Definition 1 [22] *A point \bar{x} gives a tilt stable local minimum of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ if $f(\bar{x})$ is finite and there exists an $\varepsilon > 0$ such that the mapping*

$$m : v \mapsto \operatorname{argmin}_{\|x - \bar{x}\| \leq \varepsilon} \{f(x) - \langle x, v \rangle\} \quad (1)$$

is single valued and Lipschitz on some neighbourhood of 0 with $m(0) = \bar{x}$.

In [22] a criterion for tilt stability was given in terms of second order construction based on the coderivative of the proximal subdifferential. For any multi-function $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we denote its graph by $\operatorname{Graph} F := \{(x, y) \mid y \in F(x)\}$ and the *indicator function* $\delta_{\operatorname{Graph} F}(x, y)$ to be zero if $(x, y) \in \operatorname{Graph} F$ and $+\infty$ otherwise. The *Mordukhovich coderivative* is defined as

$$D^*F(x \mid y)(w) := \{p \in \mathbb{R}^n \mid (p, -w) \in \partial \delta_{\operatorname{Graph} F}(x, y) := N_{\operatorname{Graph} F}(x, y)\}.$$

Assume the first-order condition $0 \in \partial f(\bar{x})$ holds. In [22] the second order sufficiency condition

$$\forall \|h\| = 1, p \in D^*(\partial f)(\bar{x} | 0)(h) \text{ we have } \langle p, h \rangle \geq \beta > 0 \quad (2)$$

is studied and shown to imply a tilt-stable local minimum when f is both prox-regular at \bar{x} for $\bar{z} \in \partial f(\bar{x})$, in the sense of Rockafellar and Poliquin and subdifferentially continuous [21]. As is observed in [5] and later in [3] that another characterisation of tilt stability for this class of functions is the existence of stable strong local minimizers. Such optimality conditions have been studied in [22, 9, 5, 4]. In [7, 5] it is shown that second order information provided by the coderivative is closely related to another second order condition framed in terms of the limiting subhessian [20, 8, 7]. Denote by $S_p(f)$ the points in the domain of f at which $\partial_p f(x) \neq \emptyset$. Let $\mathcal{S}(n)$ denote the set of symmetric $n \times n$ matrices (endowed with the Frobenius norm and inner product) for which $\langle Q, hh^T \rangle = h^T Q h$. Denote the cone of positive semi-definite matrices by $\mathcal{P}(n)$.

Definition 2 Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a closed lower semi-continuous function.

1. The function f is said to be twice sub-differentiable (or possess a subjet) at x if the following set is nonempty;

$$\partial^{2,-} f(x) = \{(\nabla \varphi(x), \nabla^2 \varphi(x)) : f - \varphi \text{ has a local minimum at } x \text{ with } \varphi \in \mathcal{C}^2(\mathbb{R}^n)\}.$$

The subhessians at $(x, z) \in \text{graph } \partial f$ are given by $\partial^{2,-} f(x, z) := \{Q \in \mathcal{S}(n) \mid (z, Q) \in \partial^{2,-} f(x)\}.$

2. The limiting subjet of f at x is defined to be: $\underline{\partial}^2 f(x) = \limsup_{u \rightarrow x} \partial^{2,-} f(u)$ and the associated limiting subhessians for $z \in \partial f(x)$ are $\underline{\partial}^2 f(x, z) = \{Q \in \mathcal{S}(n) \mid (z, Q) \in \underline{\partial}^2 f(x)\}.$

3. We define the rank one barrier cone for $\underline{\partial}^2 f(x, z)$ as

$$b^1(\underline{\partial}^2 f(x, z)) := \{h \in \mathbb{R}^n \mid q(\underline{\partial}^2 f(x, z))(h) := \sup \{ \langle Qh, h \rangle \mid Q \in \underline{\partial}^2 f(x, z) \} < \infty\}.$$

We are interested in the situation when $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(x, z))$ is a linear subspace and show that when f is quadratically minorized and is prox-regular at \bar{x} for $\bar{z} \in \partial f(\bar{x})$, then \mathcal{U}^2 is indeed a linear subspace. To complete our introductory definitions we define the $\mathcal{V}\mathcal{U}$ decomposition [15]. Denote the convex hull of a set $C \subseteq \mathbb{R}^n$ by $\text{co } C$. The convex hull of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is denoted by $\text{co } f$ and corresponds to the proper lower-semi-continuous function whose epigraph is given by $\overline{\text{co epi } f}$. When $\text{rel-int co } \partial f(\bar{x}) \neq \emptyset$ we can take $\bar{z} \in (\text{rel-int co } \partial f(\bar{x})) \cap \partial f(\bar{x})$ and define $\mathcal{V} := \text{span} \{ \text{co } \partial f(\bar{x}) - \bar{z} \}$ and $\mathcal{U} := \mathcal{V}^\perp$. We show that it is always the case that $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) \subseteq \mathcal{U}$, and so we call \mathcal{U}^2 the second order component of \mathcal{U} . When $\mathcal{U}^2 = \mathcal{U}$ we say that a fast-track exists at \bar{x} for $\bar{z} \in \partial f(\bar{x})$. In this paper we investigate whether the existence of a tilt stable local minimum provides extra information regarding the existence of a smooth manifold over which a smooth function interpolates the values of the f . We are able to show the following positive results. Let $h(w) := f(\bar{x} + w)$.

Theorem 3 Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function, quadratically minorized, and prox-regular at \bar{x} for $0 \in \partial f(\bar{x})$. Suppose in addition f admits a nontrivial subspace $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(\bar{x}, 0)) \subseteq \mathcal{U}$. Suppose that f has a tilt stable local minimum at \bar{x} then for $g(w) := [\text{co } h](w)$ and $\{v(u)\} = \text{argmin}_{v' \in \mathcal{V} \cap B_\varepsilon(0)} f(\bar{x} + u + v') : \mathcal{U}^2 \rightarrow \mathcal{V}^2$, we have $g(u + v(u)) = f(\bar{x} + u + v(u))$ and $\nabla_u g(u + v(u))$ existing as Lipschitz functions for $u \in B_\delta^{\mathcal{U}^2}(0)$. Moreover $\mathcal{M} := \{(u, v(u)) \mid u \in B_\varepsilon^{\mathcal{U}^2}(0)\}$ is a manifold on which the restriction to \mathcal{M} of f is smooth.

Assuming a little more we obtain the smoothness of v and in addition the smoothness of the manifold.

Theorem 4 Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function, quadratically minorized and prox-regular at \bar{x} for $0 \in \partial f(\bar{x})$. Suppose in addition that $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(\bar{x}, 0)) = \mathcal{U}$ is a linear subspace (i.e. \mathcal{U} admits a fast track) with $\{v(u)\} = \text{argmin}_{v' \in \mathcal{V} \cap B_\varepsilon(0)} f(\bar{x} + u + v') : \mathcal{U} \rightarrow \mathcal{V}$. Suppose that f has a tilt stable local minimum at \bar{x} for $0 \in [\text{rel-int co } \partial f(\bar{x})] \cap \partial f(\bar{x})$ then for $g(w) := [\text{co } h](w)$ (and $\{v(u)\} = \text{argmin}_{v' \in \mathcal{V} \cap B_\varepsilon(0)} \{g(u + v')\}$) one has g , as defined below, a $C^{1,1}(B_\delta^{\mathcal{U}}(0))$ smooth function

$$\begin{aligned} g(u + v(u)) &= f(\bar{x} + u + v(u)) \quad \text{and} \\ \nabla_w g(u + v(u)) &= (e_{\mathcal{U}}, \nabla v(u))^T \partial g(u + v(u)) \end{aligned}$$

(where $e_{\mathcal{U}}$ is the identity operator on \mathcal{U}). Moreover suppose we have a $\delta > 0$ such that for all $z_{\mathcal{V}} \in B_{\delta}(0) \cap \mathcal{V} \subseteq \text{co } \partial_{\mathcal{V}} f(\bar{x})$ we have a common

$$v(u) \in \operatorname{argmin}_{v \in \mathcal{V} \cap B_{\varepsilon}(0)} \{f(\bar{x} + u + v) - \langle z_{\mathcal{V}}, v \rangle\} \quad (3)$$

for all $u \in B_{\varepsilon}(0) \cap \mathcal{U}$ then $\mathcal{M} := \{(u, v(u)) \mid u \in B_{\varepsilon}^{\mathcal{U}}(0)\}$ is a smooth manifold on which $u \mapsto f(\bar{x} + u + v(u))$ is $C^{1,1}(B_{\delta}^{\mathcal{U}}(0))$ smooth and $u \mapsto v(u)$ is differentiable.

1.1 The $\mathcal{V}\mathcal{U}$ decomposition

Under the $\mathcal{V}\mathcal{U}$ decomposition [15] for a given $\bar{z} \in [\text{rel-int co } \partial f(\bar{x})] \cap \partial f(\bar{x})$ we have, by definition,

$$\bar{z} + B_{\varepsilon}(0) \cap \mathcal{V} \subseteq \text{co } \partial f(\bar{x}) \quad \text{for some } \varepsilon > 0. \quad (4)$$

One can then decompose $\bar{z} = \bar{z}_{\mathcal{U}} + \bar{z}_{\mathcal{V}}$ so that when $w = u + v \in \mathcal{U} \oplus \mathcal{V}$ we have $\langle \bar{z}, w \rangle = \langle \bar{z}_{\mathcal{U}}, u \rangle + \langle \bar{z}_{\mathcal{V}}, v \rangle$. Indeed we may decompose into the direct sum $\mathcal{U} \oplus \mathcal{V}$ and point $x = x_{\mathcal{U}} + x_{\mathcal{V}}$ and use the following norm for this decomposition $\|x - \bar{x}\|^2 := \|x_{\mathcal{U}} - \bar{x}_{\mathcal{U}}\|^2 + \|x_{\mathcal{V}} - \bar{x}_{\mathcal{V}}\|^2$. Denote the projection onto the subspaces \mathcal{U} and \mathcal{V} by $P_{\mathcal{U}}(\cdot)$ and $P_{\mathcal{V}}(\cdot)$, respectively. Denote $\partial_{\mathcal{V}} f(\bar{x}) := P_{\mathcal{V}}(\partial f(\bar{x}))$ and $\partial_{\mathcal{U}} f(\bar{x}) := P_{\mathcal{U}}(\partial f(\bar{x}))$. Let $\delta_C(x)$ denote the indicator function of a set C , $\delta_C(x) = 0$ iff $x \in C$ and $+\infty$ otherwise. Let f^* denote the convex conjugate of a function f .

Remark 5 The condition (4) implies $\bar{z} \in \text{rel-int co } \partial f(\bar{x})$ and thus one can take $\mathcal{V} := \text{span}\{\text{co } \partial f(\bar{x}) - \bar{z}\} = \text{affine-hull co } \partial f(\bar{x})$ which is independent of the choice of $\bar{z} \in \text{co } \partial f(\bar{x})$.

Proposition 6 Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\infty}$ is a proper lower semi-continuous function with $\partial^{\infty} f(\bar{x}) = \{0\}$.

1. We have

$$\mathcal{U} = \left\{ u \mid -\delta_{\partial f(\bar{x})}^*(-u) = \delta_{\partial f(\bar{x})}^*(u) \right\}. \quad (5)$$

2. The function

$$H(\cdot) := f(\bar{x} + \cdot) : \mathcal{U} \rightarrow \mathbb{R}_{\infty}$$

is strictly differentiable at 0 and single valued with $\partial H(0) = \{\bar{z}_{\mathcal{U}}\}$. Moreover H (as a function defined on \mathcal{U}) is continuous with H and $-H$ regular functions at 0.

3. We have

$$\partial f(\bar{x}) = \{\bar{z}_{\mathcal{U}}\} \oplus \partial_{\mathcal{V}} f(\bar{x}).$$

4. Suppose f is regular and we have $\varepsilon > 0$ such that for all $z_{\mathcal{V}} \in B_{\varepsilon}(\bar{z}_{\mathcal{V}}) \cap \mathcal{V} \subseteq \partial_{\mathcal{V}} f(\bar{x})$ there is a common

$$v(u) \in \operatorname{argmin}_{v \in \mathcal{V} \cap B_{\varepsilon}(0)} \{f(\bar{x} + u + v) - \langle z_{\mathcal{V}}, v \rangle\}$$

for all $u \in B_{\varepsilon}(0) \cap \mathcal{U}$, then we have

$$\text{cone}[\partial_{\mathcal{V}} f(\bar{x} + u + v(u)) - \bar{z}_{\mathcal{V}}] \supseteq \mathcal{V}. \quad (6)$$

Proof. (1) If $u \in \mathcal{U}$ then by construction we have

$$-\delta_{\partial f(\bar{x})}^*(-u) = -\delta_{\text{co } \partial f(\bar{x})}^*(-u) = \delta_{\text{co } \partial f(\bar{x})}^*(u) = \delta_{\partial f(\bar{x})}^*(u) \quad (7)$$

giving the containment of \mathcal{U} in the right hand side of (5). For u satisfying (7) then $\langle z - \bar{z}, u \rangle = 0$ for all $z \in \text{co } \partial f(\bar{x})$. That is, $u \perp [\text{co } \partial f(\bar{x}) - \bar{z}]$ and hence $u \perp \mathcal{V} = \mathcal{U}^{\perp}$ for all $u \in \mathcal{U}$.

(2) For $h(\cdot) := f(\bar{x} + \cdot)$ then $H = h + \delta_{\mathcal{U}}$ so $h(u) = H(u)$ when $u \in \mathcal{U}$. Then as $\partial^{\infty} f(\bar{x}) = \{0\}$, by [23, Exercise 10.10] we have

$$\partial H(0) \subseteq \partial f(\bar{x}) + N_{\mathcal{U}}(0) = \partial f(\bar{x}) + \mathcal{V}.$$

Then restricting to \mathcal{U} we have $\partial H(0)|_{\mathcal{U}} \subseteq \partial_{\mathcal{U}} f(\bar{x})$ and so

$$-\delta_{\partial f(\bar{x})}^*(-u) \leq -\hat{d}H(0)(-u) \leq \hat{d}H(0)(u) \leq \delta_{\partial f(\bar{x})}^*(u),$$

where $\hat{d}H(u) := \limsup_{x \rightarrow 0, t \downarrow 0} \inf_{u' \rightarrow u} \frac{1}{t}(f(x + tu') - f(x)) = \delta_{\partial H(0)}(u)$, implying $-\hat{d}H(0)(-u) = \hat{d}H(0)(u)$ for all $u \in \mathcal{U}$. By [23, Theorem 9.18] we have $\partial H(0)$ a singleton with H continuous at 0 and H and $-H$ regular. As $\bar{z}_{\mathcal{U}} \in \partial H(0)$ we have $\partial H(0) = \{\bar{z}_{\mathcal{U}}\}$, so $\partial_{\mathcal{U}} f(\bar{x}) = \{\bar{z}_{\mathcal{U}}\}$.

(3) Since $\partial f(\bar{x}) \subseteq \bar{z} + \mathcal{V} = \bar{z}_{\mathcal{U}} + \mathcal{V}$ we get $\partial f(\bar{x}) = \{\bar{z}_{\mathcal{U}}\} \oplus \partial_{\mathcal{V}} f(\bar{x})$. Finally we note that when $v(u) \in \operatorname{argmin}_{v \in \mathcal{V} \cap B_{\varepsilon}(0)} \{f(\bar{x} + u + v) - \langle z_{\mathcal{V}}, v \rangle\}$ for all $u \in B_{\varepsilon}(0) \cap \mathcal{U}$ and $z_{\mathcal{V}} \in B_{\varepsilon}(\bar{z}_{\mathcal{V}}) \cap \mathcal{V}$ we have, due to the necessary optimality conditions, that

$$z_{\mathcal{V}} \in \partial_{\mathcal{V}} f(\bar{x} + u + v(u))$$

and hence $B_{\varepsilon}(\bar{z}_{\mathcal{V}}) \cap \mathcal{V} \subseteq \partial_{\mathcal{V}} f(\bar{x} + u + v(u))$ giving (6). ■

2 A second order \mathcal{VU} decomposition

We will have need to discuss second order behavior in this paper and as a consequence it will be useful to define a refinement of this decomposition that takes into account second order behavior. In most treatment of the \mathcal{VU} decomposition one finds that not only by restricting f to \mathcal{U} do we find f is smooth we also find that there is a kind of regular second order behavior as well [15]. This is also often associated with smooth manifold substructures. Denote $\Delta_2 f(x, t, p, u) := 2 \frac{f(x+tu) - f(x) - t\langle p, u \rangle}{t^2}$.

Definition 7 Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\infty}$ is a closed lower semi-continuous function.

1. Denoting $S_2(f) = \{x \in \operatorname{dom}(f) \mid \nabla^2 f(x) \text{ exists}\}$, then the limiting Hessians at (\bar{x}, \bar{z}) are given by:

$$\begin{aligned} \overline{D}^2 f(\bar{x}, \bar{z}) &= \{Q \in \mathcal{S}(n) \mid Q = \lim_{n \rightarrow \infty} \nabla^2 f(x_n) \\ &\text{where } \{x_n\} \subseteq S_2(f), x_n \xrightarrow{f} \bar{x} \text{ and } \nabla f(x_n) \rightarrow \bar{z}\}. \end{aligned}$$

2. Define the second order Dini-directional derivative of f by $f''_{-}(\bar{x}, z, h) = \liminf_{t \downarrow 0, u \rightarrow h} \Delta_2 f(\bar{x}, t, z, u)$.
3. The second-order circa-derivative at \bar{x} with respect to z and h is given by

$$f^{\uparrow\uparrow}(\bar{x}, z, h) = \limsup_{(x', z') \rightarrow_{S_p(f)} (\bar{x}, z), t \downarrow 0} \inf_{u' \rightarrow h} \Delta_2 f(x', t, z', u')$$

where $(x', z') \rightarrow_{S_p(f)} (\bar{x}, z)$ means $x' \xrightarrow{f} \bar{x}$, $z' \in \partial_p f(x')$ and $z' \rightarrow z$.

Define $\partial^{2,+} f(x, z) := -\partial^{2,-}(-f)(x, -z)$ then when $Q \in \partial^{2,-} f(x, z) \cap \partial^{2,+} f(x, z)$ it follows that $Q = \nabla^2 f(x)$ and $z = \nabla f(x)$. If $f''_{-}(\bar{x}, z, h)$ is finite then $f'_{-}(\bar{x}, h) = \langle z, h \rangle$. It must be stressed that these second order objects may not exist everywhere but as $\partial^{2,-} f(x)$ is non-empty on a dense subset of its domain when f is lower semi-continuous then at worst so are the limiting objects. In finite dimensions this concept is closely related to the proximal subdifferential. The subhessian is always a closed convex set of matrices while $\underline{\partial}^2 f(\bar{x}, z)$ may not be convex (just as $\partial_p f(\bar{x})$ is convex while $\partial f(\bar{x})$ often is not).

If a function is para-concave or para-convex we have (by Aleksandrov's theorem) the set $S_2(f)$ is of full Lebesgue measure in $\operatorname{dom} f$. A function f is *para-concave* around \bar{x} when there exists a $c > 0$ and a ball $B_{\varepsilon}(\bar{x})$ within which the function $x \mapsto f(x) - \frac{c}{2} \|x\|^2$ is finite concave (conversely f is para-convex around \bar{x} iff $-f$ is para-concave around \bar{x}). A function is $C^{1,1}$ when ∇f exists and satisfies a Lipschitz property. In [7, Lemma 2.1], it is noted that f is locally $C^{1,1}$ iff f is simultaneously a locally para-convex and para-concave function. The next observation in (8) was first made in [20] and later used in [14]. The additional observations are from [7].

Proposition 8 ([7], Proposition 4.2 and 4.5) If f is lower semi-continuous then for $z \in \partial f(\bar{x})$ we have

$$\overline{D}^2 f(\bar{x}, z) - \mathcal{P}(n) \subseteq \underline{\partial}^2 f(\bar{x}, z). \quad (8)$$

If we assume in addition that f is continuous and a para-concave function around \bar{x} then equality holds in (8). In general we have

$$q(\underline{\partial}^2 f(\bar{x}, z))(h) := \sup\{\langle Q, hh^t \rangle \mid Q \in \underline{\partial}^2 f(\bar{x}, z)\} \leq f^{\uparrow\uparrow}(\bar{x}, z, h) \text{ for all } h. \quad (9)$$

When f is locally $C^{1,1}$ then (9) holds with equality.

A weakened form of para-convexity is prox-regularity. Denote the infimal convolution of f by $f_\lambda(x) := \inf_{u \in \mathbb{R}^n} (f(u) + \frac{1}{2\lambda} \|x - u\|^2)$.

Definition 9 Let the function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ be finite at \bar{x} .

1. The function f is prox-regular at \bar{x} for \bar{v} with respect to $\varepsilon > 0$ and $r \geq 0$, where $\bar{v} \in \partial f(\bar{x})$, if f is locally lower semi-continuous at \bar{x} and

$$f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{r}{2} \|x' - x\|^2$$

whenever $\|x' - \bar{x}\| \leq \varepsilon$ and $\|x - \bar{x}\| \leq \varepsilon$ and $|f(x) - f(\bar{x})| \leq \varepsilon$ with $\|v - \bar{v}\| \leq \varepsilon$ and $v \in \partial f(x)$.

2. The function f is subdifferentially continuous at \bar{x} for \bar{v} , where $\bar{v} \in \partial f(\bar{x})$, if for every $\delta > 0$ there exists $\varepsilon > 0$ such that $|f(x) - f(\bar{x})| \leq \delta$ whenever $\|x - \bar{x}\| \leq \varepsilon$ and $\|v - \bar{v}\| \leq \varepsilon$ with $v \in \partial f(x)$.

We will have need to discuss the rank-1 support $q(\mathcal{A})(h) := \sup \{ \langle Q, hh^T \rangle \mid Q \in \mathcal{A} \}$ for a subset $\mathcal{A} \subseteq \mathcal{S}(n)$, in our case $\mathcal{A} = \underline{\partial}^2 f(\bar{x}, z)$. The rank one barrier cone is $b^1(\mathcal{A}) := \{h \in \mathbb{R}^n \mid q(\mathcal{A})(h) < \infty\}$. The rank-1 support is an even, positively homogeneous degree 2 function (i.e. $q(\mathcal{A})(h) = q(\mathcal{A})(-h)$ and $q(\mathcal{A})(th) = t^2 q(\mathcal{A})(h)$) and its domain is the union of a cone C and its negative i.e.

$$\text{dom } q(\mathcal{A})(\cdot) := b^1(\mathcal{A}) = C \cup (-C). \quad (10)$$

Note also that we always have $-\mathcal{P}(n) \subseteq \underline{\partial}^2 f(\bar{x}, z)$. In the first order case we have $S(\partial f(\bar{x}), h) = f^\uparrow(\bar{x}, h)$ while $S(\partial_p f(\bar{x}), h) \leq f'_-(\bar{x}, h)$. The following was first observed in [8]

$$q(\underline{\partial}^{2,-} f(\bar{x}, z))(u) = \min \{ f''_-(\bar{x}, z, u), f''_-(\bar{x}, z, -u) \} = f''_s(\bar{x}, z, u) := \liminf_{t \rightarrow 0, u' \rightarrow u} \Delta_2 f(x, t, z, u').$$

Hence if we work with subsets we are in effect dealing with objects dual to the lower, symmetric, second-order epi-derivative. Such support functions are rather prevalent in second order analysis and is shown by the following theorem.

Theorem 10 ([8]) Let $g : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ be proper (i.e. $g(u) \neq -\infty$ anywhere) and $\text{dom } g \neq \emptyset$. For $u, v \in \mathbb{R}^n$, define $q(u, v) = \infty$ if u is not a positive scalar multiple of v or vice versa, and $q(\alpha u, u) = q(u, \alpha u) = \alpha g(u)$ for any $\alpha \geq 0$. Then q is a rank one support of a set $\mathcal{A} \subseteq \mathcal{S}(n)$ with $-\mathcal{P}(n) \subseteq 0^+ \mathcal{A}$ if and only if

1. g is positively homogeneous of degree 2.
2. g is lower semicontinuous.
3. $g(-u) = g(u)$ (symmetry).

On reflection it is clear that all second order directional derivative possess properties 1. and 3. of the above Theorem and those that are topologically well defined possess 2. as well.

Proposition 11 ([7], Corollary 6.1) Suppose that f is quadratically minorized and is prox-regular at \bar{x} for $\bar{z} \in \partial f(\bar{x})$ with respect to ε and r . Then

$$q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) = \limsup_{\lambda \downarrow 0} \inf_{h' \rightarrow h} (f - \langle \bar{z}, \cdot \rangle)_\lambda^{\uparrow\uparrow}(\bar{x}, 0, h') \quad (11)$$

and $h \mapsto q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) + r\|h\|^2$ is convex. When f is subdifferentially continuous the right hand side of (11) equals $f^{\uparrow\uparrow}(\bar{x}, \bar{z}, h)$.

Denote the recession directions of a convex set C by $0^+ C$.

Corollary 12 Suppose that f is quadratically minorized and is prox-regular at \bar{x} for $\bar{z} \in \partial f(\bar{x})$ with respect to ε and r . Then $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ is a linear subspace of \mathbb{R}^n .

Proof. Note that $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) = \text{dom}[q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(\cdot)]$ is convex under the assumption of Proposition 11. Let C be the cone given in (10) then $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) = \text{co}(C \cup (-C)) = \text{span } C$. As $\pm C \subseteq 0^+ b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ we have C a subset of the lineality space of the convex set $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$. But as (10) holds $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ coincides with its lineality space and hence is a subspace. ■

Definition 13 Let the function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ be finite at \bar{x} . When $b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ is a linear subspace of \mathbb{R}^n we call $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ the second order component of the \mathcal{U} -space.

We now justify this definition but first require the following result.

Proposition 14 ([7], Corollary 3.3) Let $\{\mathcal{A}(v)\}_{v \in W}$ be a family of non-empty rank-1 representers (i.e. $\mathcal{A}(v) \subseteq \mathcal{S}(n)$ and $-\mathcal{P}(n) \subseteq 0^+ \mathcal{A}(v)$ for all v) and W a neighbourhood of w . Suppose that $\limsup_{v \rightarrow w} \mathcal{A}(v) = \mathcal{A}(w)$. Then

$$\limsup_{v \rightarrow w} \inf_{u \rightarrow h} q(\mathcal{A}(v))(u) = q(\mathcal{A}(w))(h) \quad (12)$$

Lemma 15 Let the function $f : \mathbb{R}^n \mapsto \mathbb{R}_\infty$ be finite at \bar{x} and denote $\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$. Then

$$\mathcal{U}^2 \subseteq \mathcal{U} = \left\{ h \mid -\delta_{\partial f(\bar{x})}^*(-h) = \delta_{\partial f(\bar{x})}^*(h) = \langle \bar{z}, h \rangle \right\}. \quad (13)$$

Proof. Take $h \in \mathcal{U}^2$ so $\pm h \in b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$. As $\partial^{2,-} f(x', z') \neq \emptyset$ iff $z' \in \partial_p f(x')$ it follows via an elementary argument that $\underline{\partial}^2 f(\bar{x}, \bar{z}) = \limsup_{(x', z') \rightarrow_{S_p(f)} (\bar{x}, \bar{z})} \partial^{2,-} f(x', z')$. Then Proposition 14 gives

$$\begin{aligned} q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) &= \limsup_{(x', z') \rightarrow_{S_p(f)} (\bar{x}, \bar{z})} \inf_{u \rightarrow h} q(\partial^{2,-} f(x', z'))(u) \\ &= \sup_{\delta > 0} \limsup_{(x', z') \rightarrow_{S_p(f)} (\bar{x}, \bar{z})} \inf_{u' \in B_\delta(h)} q(\partial^{2,-} f(x', z'))(u'). \end{aligned}$$

Hence for all $h \in b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ and all $(x_k, z_k) \rightarrow_{S_p(f)} (\bar{x}, \bar{z})$ there exists $u_k \rightarrow h$ such that

$$\infty > q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) \geq q(\partial^{2,-} f(x_k, z_k))(u_k) = f_s''(x_k, z_k, u_k) := \liminf_{t \rightarrow 0, u'_k \rightarrow u_k} \Delta_2 f(x_k, t, z_k, u'_k).$$

So for each k we have $f_s''(x_k, z_k, u_k) < +\infty$, implying for all $\eta > 0$ that

$$\begin{aligned} f_-'(x_k, \pm u_k) - \langle z_k, \pm u_k \rangle &= \liminf_{t \downarrow 0, u'_k \rightarrow u_k} \frac{1}{t} [(f(x_k \pm t u'_k) - f(x_k)) - t \langle z_k, \pm u'_k \rangle] \\ &= \liminf_{t \downarrow 0, u'_k \rightarrow u_k} |t| [\Delta_2 f(x_k, \pm t, z_k, u'_k)] \\ &= \liminf_{t \rightarrow 0, u'_k \rightarrow u_k} |t| [\Delta_2 f(x_k, t, z_k, u'_k)] \leq \eta f_s''(x_k, z_k, u_k). \end{aligned}$$

Hence for all $\eta > 0$ and k large we have $f_-'(x_k, \pm u_k) \leq \langle z_k, \pm u_k \rangle + \eta$. As this holds for all $(x_k, z_k) \rightarrow_{S_p(f)} (\bar{x}, \bar{z})$ we have

$$f^\uparrow(\bar{x}, \pm h) = \limsup_{(x', z') \rightarrow_{S_p(f)} (\bar{x}, \bar{z})} \inf_{u \rightarrow \pm h} f_-'(x', u') = \min_{\{u_k \rightarrow h\}} \max_{\{(x_k, z_k) \rightarrow_{S_p(f)} (\bar{x}, \bar{z})\}} f_-'(x_k, \pm u_k) \leq \langle \bar{z}, \pm h \rangle.$$

By construction $(x_k, z_k) \rightarrow_{S_p(f)} (\bar{x}, \bar{z})$ implies $\bar{z} \in \partial f(\bar{x})$. As $f^\uparrow(\bar{x}, \pm h) = \delta_{\partial f(\bar{x})}^*(\pm h) \geq \langle \bar{z}, \pm h \rangle$ we have $\delta_{\partial f(\bar{x})}^*(h) = \langle \bar{z}, h \rangle = -\langle \bar{z}, -h \rangle = -\delta_{\partial f(\bar{x})}^*(-h)$ and so $h \in \mathcal{U}$, as desired. ■

We immediately have the following.

Corollary 16 Suppose that f is quadratically minorized and is prox-regular at \bar{x} for $\bar{z} \in \partial f(\bar{x})$ with respect to ε and r . Then there exists a second-order component $\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) \subseteq \mathcal{U}$ of the \mathcal{U} -space.

As $\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) \subseteq \mathcal{U}$ it follows from (13) and Proposition 6 part 3 that we have $\bar{z}_\mathcal{U} = z_\mathcal{U}$ for all $z \in \partial f(\bar{x})$. We finish by generalizing the notion of "fast track" [15].

Definition 17 We say f possesses a "fast track" at \bar{x} iff there exists $\bar{z} \in \partial f(\bar{x})$ for which

$$\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x}, \bar{z})) = \mathcal{U}.$$

From Proposition 8 we see that $\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x}, \bar{z}))$ provides the subspace within which the eigenvectors of the limiting Hessians remain bounded.

Lemma 18 Suppose f is quadratically minorized and prox-regular function at \bar{x} for $\bar{z} \in \partial f(\bar{x})$ which possesses a nontrivial second order component $\mathcal{U}^2 \subseteq \mathcal{U}$. Then for all $\{x_n\} \subseteq S_2(f)$, $x_n \xrightarrow{f} \bar{x}$ with $z_n \rightarrow \bar{z}$ and all $h \in \mathcal{U}^2$ there is a uniform bound $M > 0$ such that for $Q_n \in \partial^{2,-} f(x_n, z_n)$ we have

$$\langle Q_n, hh^T \rangle \leq M \|h\|^2 \quad \text{for } n \text{ sufficiently large.} \quad (14)$$

Proof. We have for all $Q \in \underline{\partial}^2 f(\bar{x}, \bar{z})$ and any $h \in \mathcal{U}^2$ that

$$\langle Q, hh^T \rangle \leq q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) < +\infty.$$

As f is prox-regular, by Proposition 11 $q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(\cdot) + r\|\cdot\|^2$ is convex and finite valued on \mathcal{U}^2 , a closed subspace and therefore is locally Lipschitz. Thus $q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(\cdot)$ is locally Lipschitz continuous on \mathcal{U}^2 . Moreover a compactness argument allows us to claim it is Lipschitz continuous on the unit ball inside the space \mathcal{U}^2 and obtains a maximum, over the unit ball restricted to the space \mathcal{U}^2 , so

$$\max_{\{h \in \mathcal{U}^2 \mid \|h\| \leq 1\}} q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) \leq 2M$$

for some $M > 0$. On dividing by $\|h\|^2$ and using the positive homogeneity of degree 2 of the rank-1 support the result follows from the inequality

$$\langle Q, hh^T \rangle \leq q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) \leq 2M \|h\|^2$$

for all $Q \in \underline{\partial}^2 f(\bar{x}, \bar{z})$ and any $h \in \mathcal{U}^2$. Now apply Proposition 14 and the continuity of the rank-1 support on \mathcal{U}^2 to obtain (14). ■

We call (for $\mathcal{A} \subseteq \mathcal{S}(n)$)

$$\mathcal{A}^1 := \{Q \in \mathcal{S}(n) \mid q(\mathcal{A})(h) \geq \langle Q, hh^T \rangle, \forall h\}$$

the symmetric rank-1 hull of $\mathcal{A} \subseteq \mathcal{S}(n)$. Note that by definition $q(\mathcal{A})(h) = q(\mathcal{A}^1)(h)$. When $\mathcal{A} = \mathcal{A}^1$, we say \mathcal{A} is a symmetric rank-1 representer. First note that if $Q \in \mathcal{A}^1$, then $Q - P \in \mathcal{A}^1$ for $P \in \mathcal{P}(n)$, the set of all positive-semidefinite forms. As usual we have denoted the indicator function of a set \mathcal{A} by $\delta_{\mathcal{A}}(Q)$ which equals zero if $Q \in \mathcal{A}$ and $+\infty$ otherwise. In general the recession directions $0^+ \mathcal{A}^1 \supseteq -\mathcal{P}(n)$. Consequently the convex support function $\delta_{\mathcal{A}^1}^*(P) := \sup \{\langle Q, P \rangle : \text{tr } QP \mid Q \in \mathcal{A}^1\} = +\infty$ if $P \notin \mathcal{P}(n)$. It is noted in [8] that $0^+ \mathcal{A}^1 = -\mathcal{P}(n)$ iff $q(\mathcal{A})(h) < +\infty$ for all h .

Lemma 19 ([6]) For any $\mathcal{A} \subseteq \mathcal{S}(n)$, then $\text{co}(\mathcal{A} - \mathcal{P}(n)) = \mathcal{A}^1$.

We can combine this observation with [23, Theorem 13.52] that gives a characterisation of the convex hull of the coderivative in terms of limiting Hessians for a $C^{1,1}$ function f .

Corollary 20 Suppose f is locally $C^{1,1}$ around x then the Mordukovich coderivative satisfies

$$\begin{aligned} \text{co } D^*(\partial f)(x|z)(h) &= \text{co}\{Ah \mid A = \lim_k \nabla^2 f(x^k) \text{ for some } x^k \in S_2(f) \rightarrow x \text{ with } \nabla f(x^k) \rightarrow z\} \\ &= \text{co} \left[\overline{D}^2 f(x, z)h \right] = \left[\text{co } \overline{D}^2 f(x, z) \right] h \subseteq \left[\left(\overline{D}^2 f(x, z) \right)^1 \right] h. \end{aligned} \quad (15)$$

and

$$\delta_{D^*(\partial f)(x|z)(h)}^*(h) = q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) = q\left(\overline{D}^2 f(x, z)\right)(h).$$

Proof. The first equality of (15) follows from [23, Theorem 13.52] and the second a restatement in terms of $\overline{D}^2 f(x, z)$. The third equality follows from preservation of convexity under a linear mapping. Clearly $\text{co } \overline{D}^2 f(x, z) \subseteq \text{co} \left[\overline{D}^2 f(x, z) - \mathcal{P}(n) \right] = \overline{D}^2 f(x, z)^1$ by Lemma 19. Moreover we must have by Proposition 8 and the linearity of $Q \mapsto \langle Q, hh^T \rangle$ that

$$\begin{aligned} q(\underline{\partial}^2 f(\bar{x}, \bar{z}))(h) &= q(\underline{\partial}^2 f(\bar{x}, \bar{z})^1)(h) = q\left(\overline{D}^2 f(x, z)^1\right)(h) = q\left(\overline{D}^2 f(x, z) - \mathcal{P}(n)\right)(h) \\ &= \sup \left\{ \langle Qh, h \rangle \mid Q \in \overline{D}^2 f(x, z) \right\} = \sup \left\{ \langle v, h \rangle \mid v \in \overline{D}^2 f(x, z)h \right\} \\ &= \sup \left\{ \langle v, h \rangle \mid v \in \text{co} \left[\overline{D}^2 f(x, z)h \right] \right\} = \sup \left\{ \langle v, h \rangle \mid v \in \text{co } D^*(f)(x|z)(h) \right\} \\ &= \sup \left\{ \langle v, h \rangle \mid v \in D^*(f)(x|z)(h) \right\} = \delta_{D^*(f)(x|z)(h)}^*(h). \end{aligned}$$

■

We finish this section reinterpreting the condition (2) for $C^{1,1}$ functions. Indeed thanks to Corollary 6 condition (2) is equivalent to the following.

Corollary 21 *If f is locally $C^{1,1}$ around x then condition (2) is equivalent to:*

$$\forall Q \in \overline{D}^2 f(x, 0) \quad \text{we have } \langle Q, hh^t \rangle \geq \beta > 0.$$

Proof. By simple convexity argument (2) is equivalent to $\langle v, h \rangle \geq \beta > 0$ for all $v \in \text{co } D^*(\partial f)(x|0)(h) = [\text{co } \overline{D}^2 f(x, 0)] h$ from which we have an equivalent condition that $\langle Qh, h \rangle \geq \beta > 0$ for all $Q \in \text{co } \overline{D}^2 f(x, 0)$. But $\langle Qh, h \rangle = \langle Q, hh^T \rangle$ (the Frobenius inner product) and linearity in Q gives $\langle Qh, h \rangle \geq \beta > 0$ for all $Q \in \overline{D}^2 f(x, 0)$ as an equivalent condition. ■

3 The localised \mathcal{U}' -Lagrangian

Then we define the localised \mathcal{U}' -Lagrangian, for any subspace $\mathcal{U}' \subseteq \mathcal{U}$, to be the function

$$L_{\mathcal{U}'}^\varepsilon(u) := \inf_{v' \in \mathcal{V}' \cap B_\varepsilon(0)} \{f(\bar{x} + u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\}$$

for some $\varepsilon > 0$, where $\mathcal{V}' := \mathcal{U}'^\perp$. Let

$$v(u) \in \text{argmin}_{v' \in \mathcal{V}' \cap B_\varepsilon(0)} \{f(\bar{x} + u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\}. \quad (16)$$

This Lagrangian differs from the modification introduced by Hare [12] in that $L_{\mathcal{U}'}^\varepsilon(\cdot)$ is locally well defined on \mathcal{U}' due to the introduction of the ball $B_\varepsilon^{\mathcal{V}'}(0) = \mathcal{V}' \cap B_\varepsilon(0)$ over which the infimum is taken. Hare assumes a quadratic minorant to justify a finite value for a sufficiently large regularization parameter used in the so-called quadratic sub-Lagrangian. Define for $u \in \mathcal{U}'$ and $v(\cdot) : \mathcal{U}' \rightarrow \mathcal{V}' \cap B_\varepsilon(0)$ the auxiliary functions

$$\begin{aligned} k_v(u) &:= h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, u + v(u) \rangle \\ \text{where } h(w) &:= f(\bar{x} + w) + \delta_{[\mathcal{U}' \cap B_\varepsilon(0)] \oplus [\mathcal{V}' \cap B_\varepsilon(0)]}(w). \end{aligned}$$

Then

$$L_{\mathcal{U}'}^\varepsilon(u) := \inf_{v' \in \mathcal{V}'} \{h(u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\}.$$

When $v(\cdot)$ is chosen as in (16) we have $L_{\mathcal{U}'}^\varepsilon(u) = k_v(u)$.

Lemma 22 *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function and assume $v(\cdot)$ is chosen as in (16). The conjugate of $k_v : \mathcal{U}' \rightarrow \mathbb{R}_\infty$ with respect to \mathcal{U}' is given by*

$$k_v^*(z_{\mathcal{U}'}) := \sup_{u \in \mathcal{U}'} \{\langle u, z_{\mathcal{U}'} \rangle - k_v(u)\} = h^*(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) = (L_{\mathcal{U}'}^\varepsilon)^*(z_{\mathcal{U}'}). \quad (17)$$

Proof. By direct calculation we have

$$\begin{aligned} k_v^*(z_{\mathcal{U}'}) &= \sup_{u \in \mathcal{U}'} \{\langle u, z_{\mathcal{U}'} \rangle - \{h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, u + v(u) \rangle\}\} \\ &= \sup_{u \in \mathcal{U}'} \left\{ \langle u, z_{\mathcal{U}'} \rangle - \min_{v' \in \mathcal{V}'} \{h(u + v') - \langle \bar{z}_{\mathcal{V}'}, u + v' \rangle\} \right\} \\ &= \sup_{(u, v') \in \mathcal{U}' \oplus \mathcal{V}'} \{\langle u + v, z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'} \rangle - h(u + v')\} = h^*(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) \end{aligned}$$

as $\langle z_{\mathcal{U}'}, v' \rangle = 0$ for all $v' \in \mathcal{V}'$. Also

$$\begin{aligned} k_v^*(z_{\mathcal{U}'}) &= \sup_{u \in \mathcal{U}'} \left\{ \langle u, z_{\mathcal{U}'} \rangle - \min_{v' \in \mathcal{V}'} \{h(u + v') - \langle \bar{z}_{\mathcal{V}'}, u + v' \rangle\} \right\} \\ &= \sup_{u \in \mathcal{U}'} \{\langle u, z_{\mathcal{U}'} \rangle - L_{\mathcal{U}'}^\varepsilon(u)\} = (L_{\mathcal{U}'}^\varepsilon)^*(z_{\mathcal{U}'}). \end{aligned}$$

■

For the rest of the paper we will assume that \bar{x} gives a tilt stable local minimum of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ and we have chosen the $\varepsilon > 0$ consistent with the tilt stability at \bar{x} and the neighbourhood system

$$B_\varepsilon^{\mathcal{U}'}(\bar{x}_{\mathcal{U}'}) \oplus B_\varepsilon^{\mathcal{V}'}(\bar{x}_{\mathcal{V}'}) := \{(x_{\mathcal{U}'}, x_{\mathcal{V}'}) \in \mathcal{U}' \oplus \mathcal{V}' \mid \|x_{\mathcal{U}'} - \bar{x}_{\mathcal{U}'}\| < \varepsilon \text{ and } \|x_{\mathcal{V}'} - \bar{x}_{\mathcal{V}'}\| < \varepsilon\}.$$

We will rely on the results of [3]. From definition 1 we have on $B_\varepsilon^{\mathcal{U}'}(\bar{x}_{\mathcal{U}'}) \oplus B_\varepsilon^{\mathcal{V}'}(\bar{x}_{\mathcal{V}'})$ that

$$f(x) \geq f(m(v)) + \langle x - m(v), v \rangle$$

where $m(\cdot)$ is as defined in (1), and hence

$$\text{co } f(x) \geq f(m(v)) + \langle x - m(v), v \rangle.$$

This observation leads to the following minor rewording of the result from [3]. It shows that there is a strong convexification process involved with tilt stability.

Proposition 23 [3, Proposition 2.6] *Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function and suppose that \bar{x} give a tilt stable local minimum of f . Then for all sufficiently small $\varepsilon > 0$, in terms of the function $h(w) := f(\bar{x} + w) + \delta_{B_\varepsilon^{\mathcal{U}'}(0) \oplus B_\varepsilon^{\mathcal{V}'}(0)}(w)$ we have*

$$\text{argmin}_{x \in B_\varepsilon^{\mathcal{U}'}(\bar{x}_{\mathcal{U}'}) \oplus B_\varepsilon^{\mathcal{V}'}(\bar{x}_{\mathcal{V}'})} [f(x) - \langle x, z \rangle] = \text{argmin}_{(u', v') \in \mathcal{U}' \oplus \mathcal{V}'} [\text{co } h(u + v) - \langle v, z \rangle]$$

for all z sufficiently close to 0. Consequently 0 is a tilt stable local minimum of $\text{co } h$.

Remark 24 In [15] it is observed that the optimality condition applied to the minimization problem that defines $L_{\mathcal{U}'}^\varepsilon$ gives rise to $\partial(\text{co } h)(u + v(u)) = \partial_{\mathcal{U}'} \text{co } h(u + v(u)) \times \{\bar{z}_{\mathcal{V}'}\}$.

We now study the subgradients of the \mathcal{U}' -Lagrangian. The next result shows that under the assumption of tilt stability we have $u := P_{\mathcal{U}'}[m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$ iff

$$z_{\mathcal{U}'} \in \partial_{\text{co}} [L_{\mathcal{U}'}^\varepsilon + \delta_{B_\varepsilon^{\mathcal{U}'}(0)}](u) \quad (18)$$

where $\partial_{\text{co}} g(u) := \{z \mid g(u') - g(u) \geq \langle z, u' - u \rangle \text{ for all } u'\}$ corresponds to the subdifferential of convex analysis.

Proposition 25 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ be a proper lower semi-continuous function with a tilt-stable local minimum at \bar{x} .*

1. *Let $u := P_{\mathcal{U}'}[m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})] \in B_\varepsilon^{\mathcal{U}'}(0)$ (where $z_{\mathcal{U}'} \in \mathcal{U}'$) then*

$$L_{\mathcal{U}'}^\varepsilon(u') - L_{\mathcal{U}'}^\varepsilon(u) \geq \langle z_{\mathcal{U}'}, u' - u \rangle \quad \text{for } u' \in B_\varepsilon^{\mathcal{U}'}(0). \quad (19)$$

Moreover $L_{\mathcal{U}'}^\varepsilon(u) = \min_{v' \in \mathcal{V}'} [\text{co } h(u + v') - \langle v', \bar{z}_{\mathcal{V}'} \rangle]$ which is attained at $v(u) = P_{\mathcal{V}'}[m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$.

2. *Conversely suppose (18) holds at any given $u \in B_\varepsilon^{\mathcal{U}'}(0)$ and let $v(u)$ be as defined in (16). Then we have $u = P_{\mathcal{U}'}[m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$ and $v(u) = P_{\mathcal{V}'}[m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})] \in B_\varepsilon^{\mathcal{V}'}(0)$.*

Proof. Consider the first case. We have $z = z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}$, where only the \mathcal{U}' component varies. The following minimum attained at the unique point $m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})$ that uniquely determines the value of $u \in \mathcal{U}'$:

$$\begin{aligned} u + v(u) &:= m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) \\ &= \text{argmin}_{x \in B_\varepsilon^{\mathcal{U}'}(\bar{x}_{\mathcal{U}'}) \oplus B_\varepsilon^{\mathcal{V}'}(\bar{x}_{\mathcal{V}'})} [f(x) - \langle x, z \rangle] \\ &= \text{argmin}_{(u', v') \in B_\varepsilon^{\mathcal{U}'}(0) \oplus B_\varepsilon^{\mathcal{V}'}(0)} [h(u' + v') - \langle v', \bar{z}_{\mathcal{V}'} \rangle - \langle u', z_{\mathcal{U}'} \rangle] \\ &= \text{argmin}_{u' \in B_\varepsilon^{\mathcal{U}'}(0)} \left[\min_{v' \in \mathcal{V}'} [h(u' + v') - \langle v', \bar{z}_{\mathcal{V}'} \rangle] - \langle u', z_{\mathcal{U}'} \rangle \right], \end{aligned} \quad (20)$$

where $u = P_{\mathcal{U}'} [m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$ and $v(u) := P_{\mathcal{V}'} [m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$. The objective value on this minimization problem equals

$$\min_{u' \in B_{\varepsilon}^{\mathcal{U}'}(0)} [L_{\mathcal{U}'}^{\varepsilon}(u') - \langle u', z_{\mathcal{U}'} \rangle] = L_{\mathcal{U}'}^{\varepsilon}(u) - \langle u, z_{\mathcal{U}'} \rangle. \quad (21)$$

Moreover by Proposition 23 we have

$$L_{\mathcal{U}'}^{\varepsilon}(u) = \min_{v' \in \mathcal{V}'} [h(u + v') - \langle v', \bar{z}_{\mathcal{V}'} \rangle] = \min_{v' \in \mathcal{V}'} [\text{co } h(u + v') - \langle v', \bar{z}_{\mathcal{V}'} \rangle].$$

For the second part we note that (18) is equivalent to (19) and hence equivalent to the identity (21), which affirms that the minimizer in the \mathcal{U}' space is attained at u and thus the minimizer in the \mathcal{V}' space in the definition of $L_{\mathcal{U}'}^{\varepsilon}(u)$ is attained at $v(u)$. This in turn can be equivalently written as (20) which affirms that $u = P_{\mathcal{U}'} [m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})]$ and $v(u) = P_{\mathcal{V}'} [m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})] \in B_{\varepsilon}^{\mathcal{V}'}(0)$. ■

Existence of convex subgradients indicates a hidden convexity.

Lemma 26 Consider $h : \mathcal{U}' \rightarrow \mathbb{R}_{\infty}$ is a proper lower semi-continuous function. Then

$$\partial_{\text{co}} h(u) \subseteq \partial[\text{co } h](u).$$

When $\partial_{\text{co}} h(u) \neq \emptyset$ then $\text{co } h(u) = h(u)$ and we have $\partial_{\text{co}} h(u) = \partial[\text{co } h](u) \subseteq \partial_p h(u)$. In particular when h is differentiable we have $\nabla h(u) = \nabla(\text{co } h)(u)$.

Proof. If $z_{\mathcal{U}} \in \partial_{\text{co}} h(u)$ then

$$\begin{aligned} h(u') - h(u) &\geq \langle z_{\mathcal{U}'}, u' - u \rangle \quad \text{for all } u' \in \mathcal{U}' \\ \text{hence } \text{co } h(u') &\geq h(u) + \langle z_{\mathcal{U}'}, u' - u \rangle \end{aligned} \quad (22)$$

and so for $u' = u$ we have $\text{co } h(u) \geq h(u) \geq \text{co } h(u)$ giving equality. Thus

$$\text{co } h(u') - \text{co } h(u) \geq \langle z_{\mathcal{U}'}, u' - u \rangle \quad \text{for all } u' \in \mathcal{U}'. \quad (23)$$

Hence $\partial_{\text{co}} h(u) \subseteq \partial[\text{co } h](u)$. When $\partial_{\text{co}} h(u) \neq \emptyset$ then $\text{co } h(u) = h(u)$ and (23) gives (22) as $h(u') \geq \text{co } h(u')$ is always true. ■

Even without the assumption of tilt stability we have the following.

Proposition 27 Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\infty}$ is a proper lower semi-continuous function and

$$v(u) \in \text{argmin}_{v' \in \mathcal{V}' \cap B_{\varepsilon}(0)} \{f(\bar{x} + u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\} : \mathcal{U}' \cap B_{\varepsilon}(0) \rightarrow \mathcal{V}'.$$

Then when $z_{\mathcal{U}'} \in \partial_{\text{co}} [L_{\mathcal{U}'}^{\varepsilon} + \delta_{B_{\varepsilon}^{\mathcal{U}'}(0)}](u)$ we have for $g(w) := \text{co } h(w)$ that

$$(u, v(u)) \in m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) = \text{argmin} \{g(u + v) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v \rangle\} \quad \text{for all } u \in B_{\varepsilon}^{\mathcal{U}'}(0). \quad (24)$$

Proof. Using inequality (19) we have for any $u' \in B_{\varepsilon}^{\mathcal{U}'}(0)$ that

$$\begin{aligned} L_{\mathcal{U}'}^{\varepsilon}(u') &\geq L_{\mathcal{U}'}^{\varepsilon}(u) + \langle z_{\mathcal{U}'}, u' - u \rangle \\ &= \inf_{v' \in \mathcal{V}'} \{h(u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\} + \langle z_{\mathcal{U}'}, u' - u \rangle \\ &= \{h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, v(u) \rangle\} + \langle z_{\mathcal{U}'}, u' - u \rangle \\ &= h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle + \langle z_{\mathcal{U}'}, u' \rangle. \end{aligned}$$

Hence for all $v' \in \mathcal{V}'$ we have

$$\begin{aligned} h(u' + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle &\geq L_{\mathcal{U}'}^{\varepsilon}(u') \\ &\geq h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle + \langle z_{\mathcal{U}'}, u' \rangle \end{aligned}$$

or for all $(u', v') \in B_{\varepsilon}^{\mathcal{U}'}(0) \oplus \mathcal{V}'$ (using orthogonality of the spaces), we have

$$h(u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle \geq h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle. \quad (25)$$

That is $(u, v(u)) \in m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'})$ and we may now apply Proposition 23. ■

The following will be required later in the paper. In part we follow a similar line of argument as in [20]. Denote $\nabla_{\mathcal{U}'}^2(\text{co } h)^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'}) = P_{\mathcal{U}'}^T \nabla^2(\text{co } h)^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'}) P_{\mathcal{U}'}$ and $\partial_{\mathcal{U}'}^{2,-}(\text{co } h)(u + v(u), z_{\mathcal{U}'} + 0_{\mathcal{V}'}) = P_{\mathcal{U}'}^T \partial^{2,-}(\text{co } h)(u + v(u), z_{\mathcal{U}'} + 0_{\mathcal{V}'}) P_{\mathcal{U}'}$. In the following we shall use the alternate notation $z_{\mathcal{U}'} + z_{\mathcal{V}'} = (z_{\mathcal{U}'}, z_{\mathcal{V}'})$ to contain the notational burden of the later.

Proposition 28 Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function and suppose that \bar{x} is a tilt stable local minimum of f . In addition suppose $0 \in [\text{rel-int co } \partial f(\bar{x})] \cap \partial f(\bar{x})$ and $v(u) \in \text{argmin}_{v' \in \mathcal{V}' \cap B_\varepsilon(0)} f(\bar{x} + u + v')$ and either $u = P_{\mathcal{U}'}[m(z_{\mathcal{U}'})] \in B_\varepsilon^{\mathcal{U}'}(0)$ or $z_{\mathcal{U}'} \in \partial L_{\mathcal{U}'}^\varepsilon(u) = \partial k_v(u)$ where $k_v(u) := h(u + v(u))$. Suppose $k_v^* : \mathcal{U}' \rightarrow \mathbb{R}_\infty$ is a $C^{1,1}(B_\varepsilon(0))$ function for some $\varepsilon > 0$ with $\nabla^2 k_v^*(z_{\mathcal{U}'})$ existing as a positive definite form. Then for $u := \nabla k_v^*(z_{\mathcal{U}'})$ we have

$$Q = \nabla^2 k_v^*(z_{\mathcal{U}'}) = \nabla_{\mathcal{U}'}^2 (\text{co } h)^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'}) \implies Q^{-1} \in \partial_{\mathcal{U}'}^{2,-} (\text{co } h)(u + v(u), z_{\mathcal{U}'} + 0_{\mathcal{V}'}).$$

Proof. We first note that by (17) we have $k_v^*(z_{\mathcal{U}'}) = h^*(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) = h^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'}) = (\text{co } h)^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'})$ and so the identities $\nabla^2 k_v^*(z_{\mathcal{U}'}) = \nabla_{\mathcal{U}'}^2 h^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'})$ and $\nabla k_v^*(z_{\mathcal{U}'}) = \nabla_{\mathcal{U}'} (\text{co } h)^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'})$ immediately follow. As $k_v(u) := h(u + v(u)) = \text{co } h(u + v(u))$ we have

$$k_v(u) + k_v^*(z_{\mathcal{U}'}) = \langle z_{\mathcal{U}'}, u \rangle = \text{co } h(u + v(u)) + h^*(z_{\mathcal{U}'} + 0_{\mathcal{V}'})$$

and hence $z_{\mathcal{U}'} \in \partial \text{co } h(u + v(u)) = \partial_{\text{co } h}(u + v(u))$ for $v(u) \in \text{argmin}_{v' \in \mathcal{V}' \cap B_\varepsilon(0)} \{\partial f(\bar{x} + u + v')\}$. Taking into account Remark 24 we have for all $u \in \mathcal{U}'$ that

$$z_{\mathcal{U}'} + 0_{\mathcal{V}'} \in \partial \text{co } h(u + v(u)) = \partial_{\mathcal{U}'} \text{co } h(u + v(u)) \times \{0_{\mathcal{V}'}\}.$$

We first show that $\begin{pmatrix} Q^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a subhessian of $\text{co } h$ at $(u, v(u))$ for $z_{\mathcal{U}'} + 0_{\mathcal{V}'} \in \partial \text{co } h(u + v(u))$ and hence $Q^{-1} \in \partial_{\mathcal{U}'}^{2,-} (\text{co } h)(u + v(u), z_{\mathcal{U}'} + 0_{\mathcal{V}'})$. Expanding via a second order Taylor expansion we have for all $z'_{\mathcal{U}'} - z_{\mathcal{U}'} \in B_\varepsilon(0)$ a function $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ with

$$\begin{aligned} (\text{co } h)^*(z'_{\mathcal{U}'} + 0_{\mathcal{V}'}) &= k_v^*(z'_{\mathcal{U}'}) \\ &= k_v^*(z_{\mathcal{U}'}) + \langle z'_{\mathcal{U}'} - z_{\mathcal{U}'}, u \rangle + \frac{1}{2} \langle Q(z'_{\mathcal{U}'} - z_{\mathcal{U}'}), (z'_{\mathcal{U}'} - z_{\mathcal{U}'}) \rangle + o(\|(z'_{\mathcal{U}'} - z_{\mathcal{U}'})\|^2) \\ &\leq (\text{co } h)^*((z_{\mathcal{U}'}, 0_{\mathcal{V}'})) + \langle z'_{\mathcal{U}'} - z_{\mathcal{U}'} + 0_{\mathcal{V}'}, u + v(u) \rangle \\ &\quad + \frac{1}{2} \left\langle \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} ((z'_{\mathcal{U}'}, 0_{\mathcal{V}'}) - (z_{\mathcal{U}'}, 0_{\mathcal{V}'})), ((z'_{\mathcal{U}'}, 0_{\mathcal{V}'}) - (z_{\mathcal{U}'}, 0_{\mathcal{V}'})) \right\rangle \\ &\quad + \delta(\varepsilon) \|((z'_{\mathcal{U}'}, 0_{\mathcal{V}'}) - (z_{\mathcal{U}'}, 0_{\mathcal{V}'}))\|^2 \end{aligned}$$

Then as $\text{co } h(u + v(u)) = \langle z_{\mathcal{U}'}, u \rangle - (\text{co } h)^*((z_{\mathcal{U}'}, 0_{\mathcal{V}'}))$ and $\langle Q(z'_{\mathcal{U}'} - z_{\mathcal{U}'}), (z'_{\mathcal{U}'} - z_{\mathcal{U}'}) \rangle \geq \alpha \|z'_{\mathcal{U}'} - z_{\mathcal{U}'}\|^2$ we have

$$\begin{aligned} \text{co } h(u' + v(u')) &= \sup_{(z'_{\mathcal{U}'}, z'_{\mathcal{V}'})} \{ \langle (z'_{\mathcal{U}'}, z'_{\mathcal{V}'}), (u', v(u')) \rangle - (\text{co } h)^*(z'_{\mathcal{U}'} + z'_{\mathcal{V}'}) \} \\ &\geq \sup_{z'_{\mathcal{U}'}} \{ \langle (z'_{\mathcal{U}'}, 0_{\mathcal{V}'}), (u', v(u')) \rangle - (\text{co } h)^*(z'_{\mathcal{U}'} + 0_{\mathcal{V}'}) \} \\ &\geq \text{co } h(u + v(u)) + \langle z_{\mathcal{U}'}, u' - u \rangle + \sup_{z'_{\mathcal{U}'} - z_{\mathcal{U}'} \in B_\varepsilon(0)} \{ \langle z'_{\mathcal{U}'} - z_{\mathcal{U}'}, u' - u \rangle \\ &\quad - \frac{1}{2} (1 + \alpha^{-1} \delta(\varepsilon)) \langle Q(z'_{\mathcal{U}'} - z_{\mathcal{U}'}), (z'_{\mathcal{U}'} - z_{\mathcal{U}'}) \rangle \} \end{aligned}$$

and when $u' - u \in (1 + \alpha^{-1} \delta(\varepsilon)) \varepsilon \|Q^{-1}\|^{-1} B_1(0)$ we have the supremum attained at

$$z'_{\mathcal{U}'} - z_{\mathcal{U}'} = (1 + \alpha^{-1} \delta(\varepsilon))^{-1} Q^{-1} (u' - u) \in B_\varepsilon(0).$$

Hence when $(u', v') \in B_{\gamma(\varepsilon)}(0)$, for $\gamma(\varepsilon) := (1 + \alpha^{-1} \delta(\varepsilon)) \varepsilon \|Q^{-1}\|^{-1}$, we have

$$\begin{aligned} \text{co } h(u' + v') &\geq \text{co } h(u' + v(u')) \geq \text{co } h(u + v(u)) + \langle (z_{\mathcal{U}'} + 0_{\mathcal{V}'}) , (u', v') - (u, v(u)) \rangle \\ &\quad + \frac{1}{2} \left\langle \begin{pmatrix} Q^{-1} & 0 \\ 0 & 0 \end{pmatrix} (u', v') - (u, v(u)), (u', v') - (u, v(u)) \right\rangle \\ &\quad - \beta(\varepsilon) \|(u', v') - (u, v(u))\|^2 \end{aligned}$$

where $\beta(\varepsilon) = [1 - (1 + \alpha^{-1} \delta(\varepsilon))^{-1}] \|Q^{-1}\| \rightarrow 0$ as $\varepsilon \rightarrow 0$. That is

$$\begin{pmatrix} Q^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \partial^{2,-} h(u + v(u), z_{\mathcal{U}'} + 0_{\mathcal{V}'}).$$

■

4 The Main Result

The main tools we use to establish our results are the convexification that tilt stable local minimum enable us to utilise [3], the correspondence between tilt stability and the strong metric regularity of the locally restricted inverse of the subdifferential and the connection conjugacy has to inversion of subdifferentials of convex functions [3] and [1]. These tools and the coderivative characterisation (2) of tilt stability (being applicable to convex functions) allows a chain of implications to be forged. The differentiability properties we seek may be deduced via strong metric regularity or alternatively via the results of [2] after invoking the Mordukhovich coderivative criteria for the Aubin property for the associated subdifferential.

In the following we repeatedly use the fact that when a function has a supporting tangent plane to its epigraph one can take the convex closure of the epigraph and the resultant set will remain entirely to that same side of that tangent hyperplane. This will be true for partial convexifications as convex combinations cannot violate the bounding plane. Once again we will consider subspaces $\mathcal{U}' \subseteq \mathcal{U}$.

Proposition 29 *Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function and $v(u) \in \operatorname{argmin}_{v' \in \mathcal{V}' \cap B_\varepsilon(0)} \{f(\bar{x} + u + v') - \langle \bar{z}_{\mathcal{V}'}, v' \rangle\}$. Then when $z_{\mathcal{U}'} \in \partial_{\operatorname{co}} [L_{\mathcal{U}'}^\varepsilon + \delta_{B_\varepsilon^{\mathcal{U}'}(0)}](u)$ we have*

$$k_v^*(z) + k_v(u) = \langle z_{\mathcal{U}'}, u \rangle$$

where $k_v(u) := h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, u + v(u) \rangle$ i.e. $z_{\mathcal{U}'} \in \partial_{\operatorname{co}} k_v(u)$ and in particular $\bar{z}_{\mathcal{U}'} \in \partial_{\operatorname{co}} k_v(u) = \partial \operatorname{co} k_v(u)$ and $k_v(u) = \operatorname{co} k_v(u)$. Moreover for $u \in \mathcal{U}'$ we have

$$\begin{aligned} k_v(u) &= [\operatorname{co} h](u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, v(u) \rangle \\ &= h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, v(u) \rangle = \operatorname{co} k_v(u), \end{aligned} \quad (26)$$

$$\text{so } h(u + v(u)) = [\operatorname{co} h](u + v(u)). \quad (27)$$

Proof. By (25) we have

$$h(u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle \geq h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle. \quad (28)$$

So $z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'} \in \partial_{\operatorname{co}} h(u + v(u)) \neq \emptyset$ by Lemma 26 we have $\operatorname{co} h(u + v(u)) = h(u + v(u))$. Hence

$$\begin{aligned} h(u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle &\geq [\operatorname{co} h](u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle \\ &\geq h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle \end{aligned}$$

On placing $v' = v(u')$ we have $h(u + v(u)) = [\operatorname{co} h](u + v(u))$ when $u' = u$ and otherwise

$$k_v(u') - \langle z_{\mathcal{U}'}, u' + v(u') \rangle \geq k_v(u) - \langle z_{\mathcal{U}'}, u + v(u) \rangle$$

or by orthogonality we have for all $u' \in \mathcal{U}'$ that

$$k_v(u') - \langle z_{\mathcal{U}'}, u' \rangle \geq k_v(u) - \langle z_{\mathcal{U}'}, u \rangle.$$

Hence $-k_v^*(z_{\mathcal{U}'}) \geq k_v(u) - \langle z_{\mathcal{U}'}, u \rangle$ implying $\langle z_{\mathcal{U}'}, u \rangle \geq k_v(u) + k_v^*(z_{\mathcal{U}'})$. The Fenchel inequality gives the results $z_{\mathcal{U}'} \in \partial_{\operatorname{co}} k_v(u) = \partial \operatorname{co} k_v(u)$ and $k_v(u) = \operatorname{co} k_v(u)$ follows from Lemma 26.

Moreover we have from (28) that

$$\begin{aligned} h(u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle &\geq h(u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle \\ &= [\operatorname{co} h](u + v(u)) - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u + v(u) \rangle \\ &\geq \operatorname{co} k_v(u) - \langle z_{\mathcal{U}'}, u \rangle \end{aligned}$$

and hence (using orthogonality)

$$\begin{aligned} [\operatorname{co} h](u' + v') - \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v' \rangle &\geq k_v(u) - \langle z_{\mathcal{U}'}, u \rangle \\ &= \{[\operatorname{co} h](u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, v(u) \rangle\} - \langle z_{\mathcal{U}'}, u \rangle \\ &\geq \operatorname{co} k_v(u) - \langle z_{\mathcal{U}'}, u \rangle. \end{aligned}$$

On placing $v' = v(u')$ we have

$$\begin{aligned} [\text{co } h](u' + v(u')) &= \langle z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}, u' + v(u') \rangle \\ &\geq [\text{co } h](u' + v(u')) - \langle \bar{z}_{\mathcal{V}'}, v(\cdot) \rangle(u') - \langle z_{\mathcal{U}'}, u \rangle \\ &\geq k_v(u) - \langle z_{\mathcal{U}'}, u \rangle \geq \text{co } k_v(u) - \langle z_{\mathcal{U}'}, u \rangle. \end{aligned}$$

and $u' = u$ and using the identities $k_v(u) = \text{co } k_v(u)$ and $[\text{co } h](u + v(u)) = h(u + v(u))$ for $u \in \mathcal{U}'$ we have (26). ■

We may now show that tilt stability is inherited by k_v .

Proposition 30 *Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function and $v(u) \in \text{argmin}_{v' \in \mathcal{V} \cap B_\varepsilon(0)} f(\bar{x} + u + v')$. Suppose that f has a tilt stable local minimum at \bar{x} for $0 \in \partial f(\bar{x})$ then $v(\cdot) : \mathcal{U}' \rightarrow \mathcal{V}'$ is uniquely defined and the associated function $k_v(\cdot) : \mathcal{U}' \rightarrow \mathbb{R}_\infty$ has a tilt stable local minimum at 0.*

Proof. In this case we have $(\bar{z}_{\mathcal{U}}, \bar{z}_{\mathcal{V}}) = (0, 0)$. By tilt stability we have $m(\cdot)$ a single valued Lipschitz functions and hence $v(\cdot)$ is unique. From Proposition 25 and $\{u\} = P_{\mathcal{U}'}[m(z_{\mathcal{U}'})]$ we have $z_{\mathcal{U}'} \in \partial_{\text{co}} [L_{\mathcal{U}'}^\varepsilon + \delta_{B_{\mathcal{U}'}^\varepsilon(0)}](u)$ and from Propositions 27 and 23 that

$$\begin{aligned} \{(u, v(u))\} &= m(z_{\mathcal{U}'} + \bar{z}_{\mathcal{V}'}) = \text{argmin}_{(u', v')} \{g(u' + v') - \langle z_{\mathcal{U}'}, u' + v' \rangle\} \\ &= \text{argmin}_{(u', v')} \{h(u' + v') - \langle z_{\mathcal{U}'}, u' + v' \rangle\} \\ \text{and so } \{u\} &= \text{argmin}_{u' \in \mathcal{U}'} \{[h(u' + v(u')) - \langle 0, u' + v(u') \rangle] - \langle z_{\mathcal{U}'}, u' \rangle\} \\ &= \text{argmin}_{u' \in \mathcal{U}'} \{k_v(u') - \langle z_{\mathcal{U}'}, u' \rangle\} \end{aligned}$$

implying $\{u\} = P_{\mathcal{U}'}[m(z_{\mathcal{U}'} + 0)] \subseteq \text{argmin}_{u' \in \mathcal{U}'} \{k_v(u') - \langle z_{\mathcal{U}'}, u' \rangle\} = \{u\}$. Hence

$$\text{argmin}_{u' \in \mathcal{U}'} \{k_v(u') - \langle z_{\mathcal{U}'}, u' \rangle\} = P_{\mathcal{U}'}[m(z_{\mathcal{U}'})]$$

is clearly a single valued, locally Lipschitz function of $z_{\mathcal{U}'} \in B_{\mathcal{U}'}^{\mathcal{U}'}(0) \subseteq \mathcal{U}'$. ■

Remark 31 *Clearly Proposition 30 implies $k_v(\cdot) : \mathcal{U}^2 \rightarrow \mathbb{R}_\infty$ has a tilt stable local minimum at 0 relative to $\mathcal{U}^2 \subseteq \mathcal{U}$.*

The following will help connect the positive definiteness of the densely defined Hessians of the convexification h with the associated uniform local strong convexity of f . This earlier results [5] may be compared with Theorem 3.3 of [3] in that it links "stable strong local minimizers of f at \bar{x} " to tilt stability. We say $f_z := f - \langle z, \cdot \rangle$ has a strict local minimum order two at x' relative to $B_\delta(\bar{x}) \ni x'$ when $f_z(x) \geq f_z(x') + \beta \|x - x'\|^2$ for all $x \in B_\delta(\bar{x})$.

Theorem 32 [5, Theorem 34 and Proposition 37] *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is lower-semicontinuous, prox-bounded (i.e. minorized by a quadratic) and $0 \in \partial_p f(\bar{x})$.*

1. *Suppose there exists $\delta > 0$ and $\beta > 0$ such that for all $(x, z) \in B_\delta(\bar{x}, 0) \cap \text{Graph } \partial_p f$ the function $f - \langle z, \cdot \rangle$ has a strict local minimum order two at x in the sense that there exists $\gamma > 0$ (depending on x, y) such that for each $x' \in \bar{B}_\gamma(x)$ we have*

$$f(x') - \langle z, x' \rangle \geq f(x) - \langle z, x \rangle + \beta \|x - x'\|^2. \quad (29)$$

Then we have for all $\|w\| = 1$ and $0 \neq p \in D^(\partial_p f)(\bar{x}, 0)(w)$ that $\langle w, p \rangle \geq \beta > 0$.*

2. *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is lower semi-continuous, prox-bounded and f is both prox-regular at \bar{x} with respect to $0 \in \partial_p f(\bar{x})$ and subdifferentially continuous with*

$$\|w\| = 1 \text{ and } p \in D^*(\partial_p f)(\bar{x}, 0)(w) \Rightarrow \langle w, p \rangle > 0. \quad (30)$$

Then there exists $\bar{\delta} > 0$ such that for $0 < \delta < \bar{\delta}$ there is a dense subset \hat{S} of $B_\delta(\bar{x}, 0) \cap \text{Graph } \partial_p f$ which contains $(\bar{x}, 0)$, with the property that for every $(x, z) \in \hat{S}$ we have x a strict local minimizer

order two of the function $f - \langle z, \cdot \rangle$ in the sense that there exists a neighbourhood $B_\gamma(x)$ (with γ depending on x and z) within which (29) holds for some uniform value $\beta > 0$. Also

$$\operatorname{argmin} \{f - \langle z, \cdot \rangle + \delta_{B_\delta(\bar{x})}\} \subseteq [\partial_p (f - \langle z, \cdot \rangle + \delta_{B_\delta(\bar{x})})]^{-1}(0) = (\partial_p f)^{-1}(z) \quad (31)$$

for all $(x, z) \in \hat{S}$ and the mapping $z \mapsto (\partial_p f)^{-1}(z)$ has the Aubin Property i.e. there exists a $\delta > 0$ such that

$$(\partial_p f)^{-1}(z) \cap B_\delta(\bar{x}) \subseteq (\partial_p f)^{-1}(z') + \kappa \|z - z'\| B_1(0) \quad (32)$$

for all $z, z' \in B_\delta(0)$, where

$$\kappa = \sup \{ \|w\| \mid \|z\| \leq 1 \text{ for } z \in D^*(\partial_p f)(\bar{x}, 0)(w) \} < +\infty.$$

The Aubin property is related to differentiability via the following result.

Theorem 33 [2, Theorem 5.3] Suppose H is a Hilbert space and $f : H \mapsto \mathbb{R}_\infty$ is l.s.c., prox-regular, and subdifferentially continuous at $\bar{x} \in \operatorname{int} \operatorname{dom} \partial f$ for some $\bar{v} \in \partial f(\bar{x})$ with respect to some $R > 0$. In addition, suppose ∂f is pseudo-Lipschitz (i.e. possess the Aubin property) at a Lipschitz rate L around \bar{x} for \bar{v} with $R > 0$. Then there exists $\varepsilon > 0$ such that $\partial f(x) = \{\nabla f(x)\}$ for all $x \in B_\varepsilon(\bar{x})$ with $x \mapsto \nabla f(x)$ Lipschitz at the rate L .

Corollary 34 Under the assumption of Proposition 30 we have $z \mapsto \partial k_v^*(z)$ a single valued Lipschitz continuous mapping in some neighbourhood of 0.

Proof. We invoke Theorems 32 and 33. As $(\operatorname{co} k_v)^* = k_v^*$ and being a convex function it is prox-regular and subdifferentially continuous so $(\partial_p \operatorname{co} k_v)^{-1} = (\partial \operatorname{co} k_v)^{-1} = \partial k_v^*$ is single valued and Lipschitz continuous by Theorem 33, noting that the tilt stability supplies the Aubin property for $(\partial \operatorname{co} k_v)^{-1}$ via (32). ■

The concept of stable strong local minimizers of f at \bar{x} corresponds to the part 2 below when $S = B_\delta(\bar{x})$, not merely a dense set. Theorem 3.3 of [3] it is noted that one may take $S = B_\delta(\bar{x}, 0) \cap \operatorname{Graph} \partial f$.

Corollary 35 [5, Corollary 39] Suppose $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous, prox-bounded and f is both prox-regular at \bar{x} with respect to $0 \in \partial_p f(\bar{x})$ and subdifferentially continuous there. Then the following are equivalent

1. For all $\|w\| = 1$ and $p \in D^*(\partial_p f)(\bar{x}, 0)(w)$ we have $\langle w, p \rangle > 0$.
2. We have the existence of a $\delta > 0$ and a dense subset S (containing $(\bar{x}, 0)$) of $B_\delta(\bar{x}, 0) \cap \operatorname{Graph} \partial_p f$ with $f - \langle p, \cdot \rangle$ possessing a strict local minimum of order two at x relative to the whole neighbourhood $B_\delta(\bar{x})$ for all $(x, p) \in S$.
3. There is a tilt-stable local minimum at \bar{x} .

As is observed in section 4 of [5, Proposition 40] another condition equivalent to all of those in Corollary 35 is the following

$$f''(x, z, u) > 0 \quad \text{for all } (x, z) \in B_\delta(\bar{x}, 0) \cap \operatorname{Graph} \partial_p f, \quad (33)$$

which is motivated by the classical observation that $f''(x, z, u) > 0$ implies $f - \langle z, \cdot \rangle$ has a strict local minimum order 2 at x .

Corollary 36 Suppose $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous, prox-bounded and f is both prox-regular at \bar{x} with respect to $0 \in \partial_p f(\bar{x})$ and subdifferentially continuous there. Then the following are equivalent:

1. For all $\|w\| = 1$ and $p \in D^*(\partial_p f)(\bar{x}, 0)(w)$ we have $\langle w, p \rangle > 0$.
2. For all $\|w\| = 1$ we have $f''(x, z, w) > 0$ for all $(x, z) \in B_\delta(\bar{x}, 0) \cap \operatorname{Graph} \partial_p f$ for some $\delta > 0$.

Proof. As is shown in [5, Proposition 40] in section 4 we have we have the condition (43) and (44) of [5, Theorem 34] holding iff we have part 2 of this corollary hold. Then we may use [5, Proposition 37] to that provides the equivalence between (43) and (44) of [5, Theorem 34] and the condition 1 of this corollary. ■

Our main goal is to demonstrate that the restriction of f to the set $\mathcal{M} := \{(u, v(u)) \mid u \in \mathcal{U}^2\}$ coincides with the restriction of a $C^{1,1}$ smooth function to \mathcal{M} . Consequently we will be focusing on the case when \mathcal{U}^2 is a linear subspace and so take $\mathcal{U}' \equiv \mathcal{U}^2$ in our previous results. The next result demonstrates when there is a symmetry with respect to conjugation in the tilt stability property for the auxiliary function k_v .

Theorem 37 Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ is a proper lower semi-continuous function, which is a prox-regular function at \bar{x} for $0 \in \partial f(\bar{x})$ with a nontrivial subspace $\mathcal{U}^2 = b^1(\bar{\partial}^2 f(\bar{x}, 0)) \subseteq \mathcal{U}$. Denote $\mathcal{V}^2 = (\mathcal{U}^2)^\perp$ and let $v(u) \in \operatorname{argmin}_{v' \in \mathcal{V}^2 \cap B_\varepsilon(0)} f(\bar{x} + u + v') : \mathcal{U}^2 \rightarrow \mathcal{V}^2$. Let $k_v(u) := h(u + v(u)) : \mathcal{U}^2 \rightarrow \mathbb{R}_\infty$. Suppose also that f has a tilt stable local minimum at \bar{x} for $0 \in \partial f(\bar{x})$ then for $p \neq 0$ we have

$$\forall q \in D^*(\nabla k_v^*)(0|0)(p) \quad \text{we have} \quad \langle p, q \rangle > 0 \quad (34)$$

and hence k_v^* has a tilt stable local minimum at $0 \in \partial k_v^*(0)$.

Proof. On application of Propositions 30 and 23 we have $\operatorname{co} k_v(\cdot) : \mathcal{U}^2 \rightarrow \mathbb{R}_\infty$ possessing a tilt stable local minimum at 0. As $\operatorname{co} k_v(\cdot)$ is convex it is prox-regular at 0 for $0 \in \partial \operatorname{co} k_v(0)$ and subdifferentially continuous at 0 [23, Proposition 13.32]. Hence we may apply [22, Theorem 1.3] to obtain the equivalent condition for tilt stability. For all $q \neq 0$

$$\langle p, q \rangle > 0 \quad \text{for all} \quad p \in D^*(\partial[\operatorname{co} k_v])(0|0)(q). \quad (35)$$

Now apply Theorem 32 part 2 to deduce the existence of a neighbourhood $B_\gamma(0)$ within which (29) holds for some uniform value $\beta > 0$. Now apply Theorem 32 part 1 to deduce that $\langle p, q \rangle \geq \beta > 0$ for all (p, q) taken in (35).

From Proposition 6 part 3, Remark 24 and Lemma 26 we see that $\nabla k_v(0) = \{0\} = \nabla \operatorname{co} k_v(0)$. Then whenever $x^k \in S_2(\operatorname{co} k_v)$ with $x^k \rightarrow 0$ (as we always have $z^k = \nabla \operatorname{co} k_v(x^k) \rightarrow 0 = \nabla \operatorname{co} k_v(0)$) it follows from Corollary 36 that we have

$$(\operatorname{co} k_v)_s''(x^k, \nabla \operatorname{co} k_v(x^k), y) = \langle \nabla^2 \operatorname{co} k_v(x^k)h, h \rangle > 0 \quad \text{for all } h \in \mathcal{U}^2 \quad (36)$$

for k sufficiently large. By Alexandrov's theorem this positive definiteness of Hessians must hold on a dense subset of some neighbourhood of zero. By the choice of $v(\cdot)$ we have $k_v(u) = L_{\mathcal{U}^2}^\varepsilon(u)$ and hence we may assert that $\partial_{\operatorname{co}} L_{\mathcal{U}^2}^\varepsilon(u) = \partial \operatorname{co} k_v(u) \neq \emptyset$ in some neighbourhood of the origin in \mathcal{U}^2 .

Since $[\operatorname{co} k_v]^* = k_v^*$ and $\nabla k_v^* = [\partial \operatorname{co} k_v]^{-1}$ we may apply [23, identity 8(19)] to deduce that for $\|q\| = 1$ we have

$$-q \in D^*([\partial \operatorname{co} k_v]^{-1})(0|0)(-p) = D^*(\nabla k_v^*)(0|0)(-p).$$

Hence we can claim that for $q \neq 0$, after a sign change, that $\langle p, q \rangle = \langle -p, -q \rangle \geq \beta > 0$. We need to rule out the possibility that $0 \in D^*(\nabla k_v^*)(0|0)(p)$ for some $p \neq 0$. To this end we may use the fact that k_v^* is $C^{1,1}$ (and convex) and apply [23, Theorem 13.52] to obtain the following characterisation of the convex hull of the coderivative in terms of limiting Hessians. Denote $S_2(k_v^*) := \{x \mid \nabla^2 k_v^*(x) \text{ exists}\}$ then

$$\operatorname{co} D^*(\nabla k_v^*)(0|0)(p) = \operatorname{co}\{Ap \mid A = \lim_k \nabla^2 k_v^*(z^k) \text{ for some } z^k \in S_2(k_v^*) \rightarrow 0\}.$$

Now suppose $0 \in D^*(\nabla k_v^*)(0|0)(p)$ then there exists $A^i = \lim_k \nabla^2 k_v^*(z_i^k)$ for $z_i^k \rightarrow 0$ such that $0 = q := \sum_{i=1}^m \lambda_i A^i p \in \operatorname{co} D^*(\nabla k_v^*)(0|0)(p)$. As $p \neq 0$ we must then have $\langle p, q \rangle = p^T (\sum_{i=1}^m \lambda_i A^i) p = 0$ for a $B := \sum_{i=1}^m \lambda_i A^i$ is symmetric positive semi-definite. Now apply the duality formula for Hessians [10] to deduce that when $x_i^k := \nabla k_v^*(z_i^k)$ then $A_i^k = \nabla^2 k_v^*(z_i^k)$ iff $(A_i^k)^{-1} = \nabla^2(\operatorname{co} k_v)(x_i^k)$. The inverse $(A_i^k)^{-1}$ exists (relative to \mathcal{U}^2) due to (36).

We now apply Lemma 18 to deduce that the limiting subhessians of $h(w) := f(\bar{x} + w)$ satisfy (14). We will want to apply this bound to the limiting subhessians of $\operatorname{co} h$ at $x_i^k + v(x_i^k)$. To this end we demonstrate that $\Delta_2 h(x_i^k + v(x_i^k), (z_i^k, 0), t, w) \geq \Delta_2(\operatorname{co} h)(x_i^k + v(x_i^k), (z_i^k, 0), t, w)$ for all $t \in \mathbb{R}$ and any w . This follows from Lemma 26, Proposition 29 in that $(z_i^k, 0) \in \partial \operatorname{co} h(x_i^k + v(x_i^k)) =$

$\partial h(x_i^k + v(x_i^k)), \text{co } h(x_i^k + v(x_i^k)) = h(x_i^k + v(x_i^k))$ and $\text{co } h(u + v) \leq h(u + v)$ for all $(u, v) \in \mathcal{U}^2 \times \mathcal{V}^2$. On taking the a limit infimum for $t \rightarrow 0$ and $w \rightarrow u \in \mathcal{U}^2$ we obtain

$$\begin{aligned} q(\partial^{2,-}(\text{co } h)(x_i^k + v(x_i^k), (z_i^k, 0)))(u) &= (\text{co } h)_s''(x_i^k + v(x_i^k), (z_i^k, 0), u) \\ &\leq h''(x_i^k + v(x_i^k), (z_i^k, 0), u) = q(\partial^{2,-}h(x_i^k + v(x_i^k), (z_i^k, 0)))(u). \end{aligned}$$

Hence the bound in (14) applies to $\partial^{2,-}(\text{co } h)(x_i^k + v(x_i^k), (z_i^k, 0))$ for k large.

As $A_k^i = \nabla^2 k_v^*(z_i^k)$ by Proposition 28 we have $(A_k^i)^{-1} = (\nabla_{\mathcal{U}^2}^2 h^*(z_i^k + 0_{\mathcal{V}^2}))^{-1} \in \partial_{\mathcal{U}^2}^{2,-}(\text{co } h)(x_i^k + v(x_i^k))$ and on restricting to the \mathcal{U}^2 space and using (17), (14) and (27) we get for all $p \in \mathcal{U}^2$ that

$$\langle A_k^i, pp^T \rangle = \langle \nabla^2 k_v^*(z_i^k), pp^T \rangle = \langle \nabla_{\mathcal{U}^2}^2 h^*(z_i^k + 0_{\mathcal{V}}), pp^T \rangle = \langle [(A_k^i)^{-1}]^{-1}, pp^T \rangle \geq \frac{1}{M}.$$

Thus $\{A_k^i\}$ are uniformly positive definite. By [10] we have $(A_k^i)^{-1} = \nabla^2(\text{co } k_v)(x_i^k)$ existing at x_i^k and hence

$$(A_k^i)^{-1}u = \nabla^2(\text{co } k_v)(x_i^k)u \in D^*(\nabla \text{co } k_v)(x_i^k | z_i^k)(u) \quad \text{for all } u \in \mathcal{U}^2.$$

Then, for $u \neq 0$, by Theorem 32 we have $\langle \nabla^2(\text{co } k_v)(x_i^k)u, u \rangle \geq \frac{\beta}{2} > 0$ for k large implying $\{(A_k^i)^{-1}\}$ remain uniformly positive definite on \mathcal{U}^2 . Hence $\{A_k^i\}$ remain uniformly bounded within a neighbourhood of the origin within \mathcal{U}^2 . Thus on taking the limit we get $A^i = \lim_k A_k^i$ is positive definite and hence $B := \sum_{i=1}^m \lambda_i A^i$ is actually positive definite, a contradiction.

As k_v^* is convex and finite at 0, it is prox-regular and subdifferentially continuous at 0 for $0 \in \partial k_v^*(0)$ by [23, Proposition 13.32]. Another application of [22, Theorem 1.3] allows us to deduce that k_v^* has a tilt stable local minimum at $0 \in \nabla k_v^*(0)$. ■

We may either use the strong metric regularity property to obtain the existence of a smooth manifold or utilizes the Mordukhovich criteria for the Aubin property [23] and the results of [2] on single valuedness of the subdifferential satisfying a pseudo-Lipschitz property, namely:

Proof. [of Theorem 3] *using strong metric regularity*

Note first that that $\mathcal{U}^2 \subseteq \mathcal{U}$ implies (13) for $\bar{z} = 0$. Let $v(u) \in \arg\min_{v' \in \mathcal{V}^2 \cap B_\varepsilon(0)} f(\bar{x} + u + v')$. We apply either [22, Theorem 1.3] or [3, Theorem 3.3] that asserts that as k_v^* is prox-regular and subdifferentially continuous at 0 for $0 \in \partial k_v^*(0)$ then ∂k_v^* is strongly metric regular at $(0, 0)$. That is there exists $\varepsilon > 0$ such that

$$B_\varepsilon(0) \cap (\partial k_v^*)^{-1}(u)$$

is single valued and locally Lipschitz for $u \in \mathcal{U}^2$ sufficiently close to 0. But as $(\partial k_v^*)^{-1} = \partial k_v^{**} = \partial[\text{co } k_v]$ is a closed convex valued mapping (and hence has connected images) we must have the existence of $\delta > 0$ such that for $u \in B_\delta^{\mathcal{U}^2}(0)$ we have $\partial[\text{co } k_v](\cdot)$ a singleton locally Lipschitz mapping (giving differentiability). As $v(u) \in \arg\min_{v' \in \mathcal{V}^2 \cap B_\varepsilon(0)} \{h(u + v') - \langle \bar{z}_{\mathcal{V}^2}, v' \rangle\} : \mathcal{U}^2 \cap B_\varepsilon(0) \rightarrow \mathcal{V}^2$ we have $k_v(u) = L_{\mathcal{U}^2}^\varepsilon(u)$ for $u \in \text{int } B_\varepsilon^{\mathcal{U}^2}(0)$. Hence $\nabla \text{co } L_{\mathcal{U}^2}^\varepsilon(u) \in \partial_{\text{co}} L_{\mathcal{U}^2}^\varepsilon(u) \neq \emptyset$ and by Corollary 29 we have on \mathcal{U}^2 that $h(u + v(u)) = [\text{co } h](u + v(u))$ and hence

$$\partial[\text{co } k_v](u) = \partial[\text{co } h](u + v(u)) = \partial g(u + v(u))$$

is single valued implying $\nabla_u g(u + v(u))$ exists where $g(\cdot) := [\text{co } h](\cdot)$. ■

Proof. [of Theorem 3] *using the single valuedness of the subdifferential satisfying a pseudo-Lipschitz property.*

We now show that $D^*(\partial[\text{co } k_v])(0 | 0)(0) = \{0\}$. To this end we use (34). Indeed this implies that $q \neq 0$ for any $p \neq 0$. Conversely we have $q = 0$ implies $p = 0$ for all $q \in D^*(\nabla k_v^*)(0 | 0)(p)$ or equivalently $p \in D^*(\partial[\text{co } k_v])(0 | 0)(q)$. Hence we have $D^*(\partial[\text{co } k_v])(0 | 0)(0) = \{0\}$. Now apply the Mordukhovich criteria for the Aubin property [23, Theorem 9.40] to deduce that $\partial[\text{co } k_v]$ has the Aubin property at 0 for $0 \in \partial[\text{co } k_v](0)$. Now apply Theorem 33 to deduce that $u \mapsto \nabla[\text{co } k_v](u)$ exists a single valued Lipschitz mapping in some ball $B_\delta^{\mathcal{U}^2}(0)$ in the space \mathcal{U}^2 . We now finish the proof as before in the first version. ■

If we assume more, essentially what is needed to obtain partial smoothness we get a smooth manifold.

Proof. [of Theorem 4] First note that when we have (3) holding using f then we must (3) holding using $g := \text{co } h$. Thus by Proposition 6 part 4 have (6) holding using g (via the usual convexification argument). As $g(w) := [\text{co } h](w)$ for $w \in B_\varepsilon(0)$ we have g a regular in $B_\varepsilon(0)$. Moreover as $g(u + v(u)) = f(\bar{x} + u + v(u))$ (and $g(w) \leq f(\bar{x} + w)$ for all w) we have the regular subdifferential of g contained in that of f . As g is regular the singular subdifferential coincides with the recession directions of the regular subdifferential [23, Corollary 8.11] and so are contained in the recession direction of the regular subdifferential of f . We are thus able to write down the following inclusions

$$\partial^\infty g(\cdot)(u + v(u)) \subseteq \partial^\infty h(\cdot)(u + v(u)) = \partial^\infty f(\bar{x} + u + v(u)) = \{0\}.$$

By the tilt stability we have v a locally Lipschitz single valued mapping. Thus by the basic chain rule of subdifferential calculus we have

$$\{\nabla_u g(u + v(u))\} = (e_{\mathcal{U}} \oplus \partial v(u))^T \partial g(u \oplus v(u))$$

is a single valued Lipschitz mapping. Under the additional assumption we have via Proposition 6 part 4 that, $\text{cone}[\partial_{\mathcal{V}} g(u + v(u))] = \mathcal{V}$ with $0 \in \text{int } \partial_{\mathcal{V}} g(u + v(u))$ for $u \in B_\varepsilon(0) \cap \mathcal{U}$. As $\partial v(u) \subseteq \mathcal{V}$ it cannot be multi-valued and still have $(e_{\mathcal{U}} \oplus \partial v(u))^T \partial g(u \oplus v(u))$ single valued. This implies the limiting subdifferential $\partial v(u)$ is single valued and hence $\nabla v(u)$ exists. As all of $\nabla_u g(u + v(u))$, $v(u)$ and $g(u \oplus v(u))$ are locally Lipschitz (so $\partial g(u \oplus v(u))$ is locally bounded) and hence $(e_{\mathcal{U}} \oplus \nabla v(u))^T \partial g(u \oplus v(u))$ is single valued and locally Lipschitz. ■

The following example demonstrates the fact that even if $\partial_w g(u + v(u))$ is multi-valued we still have $(e_{\mathcal{U}}, \nabla v(u))^T \partial_w g(u + v(u))$ single valued.

Example 38 If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f = \max\{f_1, f_2\}$ where $f_1 = x_1^2 + (x_2 - 1)^2$ and $f_2 = x_2$, then $\partial_w g(u + v(u))$ is multi valued but $(e_{\mathcal{U}}, \partial v(u))^T \partial_w g(u + v(u))$ is single valued.

Using the notation in the Theorem, we put $\bar{x} = 0$, find that $\partial f(0) = \{\alpha(0, 1 - \sqrt{5}) + (1 - \alpha)(0, 1) \mid 0 \leq \alpha \leq 1\}$ so we have $\mathcal{U} = \{\alpha(1, 0) \mid \alpha \in \mathbb{R}\}$ and $\mathcal{V} = \{\alpha(0, 1) \mid \alpha \in \mathbb{R}\}$. With $\epsilon < 1/2$ then

$$v(u) = \frac{3}{2} - \frac{\sqrt{5 - 4u^2}}{2}, \quad g(u + v(u)) = f(\bar{x} + u + v(u)) = \frac{3}{2} - \frac{\sqrt{5 - 4u^2}}{2}.$$

It follows that

$$\nabla v(u) = \frac{2u}{\sqrt{5 - 4u^2}} \quad \text{and} \quad (e_{\mathcal{U}}, \partial v(u))^T = \left(1, \frac{2u}{\sqrt{5 - 4u^2}}\right)^T.$$

Now we consider $\partial_w g(u + v(u)) = \partial_w f(u + v(u))$. At $u + v(u)$, from f_1 we know

$$t_1 = (2u, 1 - \sqrt{5 - 4u^2}) = \nabla_w f_1(u + v(u))$$

and from f_2 we know

$$t_2 = (0, 1) = \nabla_w f_2(u + v(u)).$$

Thus

$$\partial_w f(u + v(u)) = \{\alpha t_1 + (1 - \alpha)t_2 \mid 0 \leq \alpha \leq 1\},$$

that is, $\partial_w g(u + v(u))$ is multi valued. However, for all such α , we have

$$(e_{\mathcal{U}}, \partial v(u))^T (\alpha t_1 + (1 - \alpha)t_2) = 2\alpha u + (1 - \alpha)\sqrt{5 - 4u^2} \frac{2u}{\sqrt{5 - 4u^2}} = \frac{2u}{\sqrt{5 - 4u^2}}.$$

Therefore $(e_{\mathcal{U}}, \partial v(u))^T \partial_w g(u + v(u))$ is single valued.

Remark 39 The function described in Theorem 4 are quite closely related to the partial smooth class introduced by Lewis [17, 16]. Lewis calls f partially smooth at x relative to a manifold \mathcal{M} iff

1. We have $f|_{\mathcal{M}}$ is smooth around x ;
2. for all points in \mathcal{M} close to x we have f is regular and has a subgradient;

3. we have $f'_-(x, h) > -f'_-(x, -h)$ for all $h \in N_{\mathcal{M}}(x)$ and
4. the subgradient mapping $w \mapsto \partial f(w)$ is continuous at x relative to \mathcal{M} .

It is not difficult to see that $\{0\} \times \mathcal{V} = N_{\mathcal{M}}(x)$. Clearly we have 1 and 3 holding for the function described in Theorem 4. As functions that are prox-regular at a point $(x, 0) \in \text{Graph } \partial f$ are not necessarily regular at x then 2 is not immediately obvious, although a subgradient must exist. By Proposition 6 the restricted function (to \mathcal{U}) is indeed regular. Moreover the "convex representative" given by $g := \text{co } h$ is regular, thanks to convexity. The potential for $w \mapsto \partial g(w)$ to be continuous at 0 is clearly bound to the need for $w_{\mathcal{V}} \mapsto \partial_{\mathcal{V}} g(w_{\mathcal{V}})$ to be continuous at 0. As $0 \in \text{int } \partial_{\mathcal{V}} g(u + v(u))$ for $u \in B_{\varepsilon}(0) \cap \mathcal{U}$ this problem may be reduced to investigating whether $u \mapsto \text{int } \partial_{\mathcal{V}} g(u + v(u))$ is lower semi-continuous at 0. This is not self evident either. So the question as to whether g is partially smooth is still open.

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